Four-boson bound states from a functional renormalisation group

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We use the functional renormalisation group to study the spectrum of three- and four-body states in bosonic systems around the unitary limit. Our effective action includes all energy-independent contact interactions in the four-atom sector and we introduce a running trimer field to eliminate couplings that involve the atom-atom-dimer channel. The results show qualitatively similar behaviour to those from exact approaches. The truncated action we use leads to overbinding of the two four-body states seen in those treatments. It also generates a third state, although only for a very narrow range of two-body scattering lengths.

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Systems where the two-body scattering is close to the unitary limit display universal features that are independent of the underlying dynamics. This physics has been realised by ultra-cold atoms in traps, where the scattering length can tuned using Feshbach resonances [1–3]. It may also be relevant to low-energy aspects of few-nucleon systems [4, 5].

A key feature of this limit for bosonic systems (and also for fermionic ones with at least three species of particle) is the Efimov effect [6, 7]. This is the appearance of an infinite tower of three-body bound states, with energies in a constant ratio. As a result the scale invariance of the unitary limit is broken leaving only a discrete symmetry where momenta are scaled by a factor of 22.7. This physics feeds through to systems with more particles. Two four-body bound states (or, strictly speaking, narrow resonances) have been found just below each three-body state [8–11]. More recently these analyses have been extended to states of up 16 bosons [12, 13].

The scaling behaviour of the three-boson system has been analysed using the renormalisation group (RG) and the Efimov effect has been shown to result from a limit cycle in the flow of the leading three-body contact interaction [3, 14–16]. However it has proved impossible so far to extend these rigorous RG approaches to systems of four or more particles. A promising alternative tool for exploring scaling in these systems is the functional renormalisation group (FRG), particularly the version based on the Legendre-transformed effective action developed by Wetterich and co-workers [17–20]. Although in principle the method is exact, practical applications rely on truncations of the action to a finite number of terms. Nonetheless, even with fairly simple truncations, it has yielded useful results across a wide range of areas [20–22].

In previous work [23], we applied the FRG to the four-boson problem, using it to determine the values of the two-body scattering length for which the four-body bound states appear at zero energy. These correspond to points at which resonant recombination occurs in trapped cold-atom systems. Our results for these scattering lengths are in good agreement with those from exact few-body calculations [9, 10, 24] and experiment [25]. That work showed that it was important to keep all zero-derivative contact terms, including ones where dimers break up into pairs of atoms [26, 27]. It also demonstrated that energy dependence in the trimer channel was needed to generate four-body bound states. That energy dependence was described through the introduction of a trimer field [26, 28].

The results of Ref. [23] were restricted to zero-energy physics; here we extend that analysis to nonzero energy in order to access the spectrum of four-body states. Our starting point is a minor modification of the effective action used in that work:

$$\begin{aligned}\Gamma_k[\psi,\psi^*,\phi,\phi^*,\chi,\chi^*] = \int \mathrm{d}^4x \Bigg[& \psi^*\left(\mathrm{i}\,\partial_0 + E_a + \frac{\nabla^2}{2m}\right)\psi + Z_d\,\phi^*\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4m}\right)\phi + Z_t\,\chi^*\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{6m}\right)\chi \\ & - u_d\phi^*\phi - u_t\chi^*\chi - \frac{g}{2}\big(\phi^*\psi\psi + \psi^*\psi^*\phi\big) - h\big(\chi^*\phi\psi + \phi^*\psi^*\chi\big) - \lambda\,\phi^*\psi^*\phi\psi \\ & - \frac{u_{dd}}{2}\big(\phi^*\phi\big)^2 - \frac{v_d}{4}\big(\phi^*\phi^*\phi\psi\psi + \phi^*\psi^*\psi^*\phi\phi\big) - \frac{w}{4}\phi^*\psi^*\psi^*\phi\psi\psi \\ & - u_{tt}\,\chi^*\psi^*\chi\psi - \frac{u_{dt}}{2}\big(\phi^*\phi^*\chi\psi + \chi^*\psi^*\phi\phi\big) - \frac{v_t}{2}\big(\phi^*\psi^*\psi^*\chi\psi + \chi^*\psi^*\phi\psi\psi\big)\Bigg]. \end{aligned} \quad (1)$$

Here ψ, ϕ and χ are the fields corresponding to atoms (A), dimers (D) and trimers (T) respectively. This action is supplemented by regulators for each field that suppress the contributions of fluctuations with momenta below some scale k. As in our previous work, we take the form suggested by Litim [34] for the regulators $R_{a,d,t}(q,k)$. The renormalisation group (RG) equation describes the evolution of the effective action as k is lowered and more and more modes are integrated out until, in the limit $k \to 0$, the physical effective action is reached.

We work with a trimer field that runs with renormalisation scale [26, 28]. This allows us to absorb the interaction terms that contain the AAD "breakup" channel, leaving only the DD and AT channels that are present in the Faddeev-Yakubovsky equations [29]. The resulting RG equation for the effective action has the form

$$\partial_k \Gamma = -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[(\partial_k \mathbf{R})\,\left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1}\right)\right] + \frac{\delta\Gamma}{\delta\chi}\cdot\partial_k\chi + \frac{\delta\Gamma}{\delta\chi^*}\cdot\partial_k\chi^*\,, \tag{2}$$

where the final terms are generated by the running of the trimer field [19, 30]. Floerchinger and Wetterich [31] have proposed a more complicated version of the flow equation for a running field, which they derive from a change of field variables in the functional integral. The version used here can be thought of as arising instead from a shift of the expansion point of the field, in order to cancel certain terms in the effective action. This is analogous to the shift in the expansion point often used to treat systems where a symmetry is spontaneously broken (see, for example, Refs. [18, 32, 33]).

Ordinary differential equations describing the running of each of the couplings are obtained by expanding both sides of Eq. (2) in powers of the fields. More details of these equations can be found in Ref. [23] (see also Refs. [26, 27]).

The one difference between the action (1) and the one in Ref. [23] is that here we add an external energy $E_a = -\gamma^2/2m$ flowing along each atom line. In effect this shifts the expansion point for the part of action describing a system of n atoms from zero energy to nE_A. The results would be independent of this expansion point if we worked with the full, untruncated effective action. In practice, some artefacts of the truncation to a set of local terms are visible in our results and provide some indication of the errors introduced by it.

Note that all the parameters are implicitly functions of the external energy E_a and that we have chosen not to subtract a term $2Z_dE_a$ from the dimer self-energy u_d. The condition for a dimer bound state is therefore $u_d = 0$ (rather than $2E_a = u_d/Z_d$). Similarly, the condition for a trimer bound state is $u_t = 0$.

We start the evolution at some large scale K where only the atom fields are dynamical ($Z_\phi = Z_\chi = 0$) and $u_d(K)$ is fixed so that $g^2/u_d(k=0)$ gives the physical atom-atom scattering length a [32, 33]. For a fixed value of the energy E_a, we scan through values of a to identify the ones at which the three-body energy, u_t, vanishes, or the four-body couplings, u_{dd} etc., diverge in the physical limit. These correspond to points at which an n-body bound state has energy nE_a.

The evolution in the three-body sector shows the periodic behaviour expected as result of the Efimov effect [6, 7]. However, the truncation of the action means that the momentum scaling factor is somewhat larger than the true value, 29.8 rather than 22.7 [26, 28].

Another consequence of truncating the energy dependence in the action is that some thresholds do not not appear in the correct places. For example, the atom-dimer threshold in the trimer propagator should be where the total energy in the three-body system is equal to the dimer binding energy, $3E_a = -1/ma^2$. Within our truncation, it appears where the atom-dimer energy $E_a + u_d/Z_d$ vanishes. Using the expressions for the full two-body self energy at $k = 0$ [35],

$$u_d(E_a,0) = \frac{g^2 m}{4\pi^2}\left(\mathrm{i}\sqrt{2mE_a} + \frac{1}{a}\right), \tag{3}$$

$$Z_d(E_a,0) = \frac{1}{2}\frac{\partial u_d}{\partial E_a} = \frac{g^2 m^2}{8\pi^2}\,\frac{\mathrm{i}}{\sqrt{2mE_a}}\,, \tag{4}$$

we find that this condition is satisfied for

$$E_a = -\frac{1}{2m}\left(\frac{4}{5a}\right)^2. \tag{5}$$

This effective threshold corresponds to a total three-body energy that differs by a factor of 24/25 from the correct one. A similar issue arises for the atom-trimer threshold in the four-body sector.

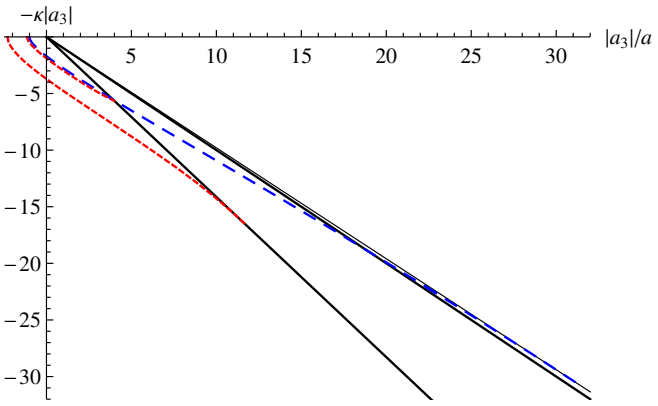

FIG. 1. Three- and four-body bound states for one Efimov cycle. The binding momenta $-\kappa|a_3|$ are plotted against the inverse atom-atom scattering length, $|a_3|/a$. The thick solid lines show the dimer bound state and the DD threshold. The long-dashed blue line shows the trimer bound state, and the short-dashed red lines the tetramer bound states. The thin solid line shows the effective AD threshold for our truncation.

As already found in our study of this system at zero energy, our approach leads to three four-body bound states in each Efimov cycle [23]. This is one more than are seen

in exact solutions of the four-body problem [8–10, 24, 36] and is likely to reflect the fact that our Efimov cycles are too long. Our results for the binding energies of the three- and four-body states in a single cycle are presented in Figs. 1 and 2. The plots show the energies, expressed in the form of binding momenta, $\kappa = \sqrt{-n\,m E_a}$, as functions of the inverse scattering length $1/a$. All quantities have been expressed in units of $1/|a_3|$, where a_3 is the atom-atom scattering length at which the three-body state becomes unbound unto three atoms.

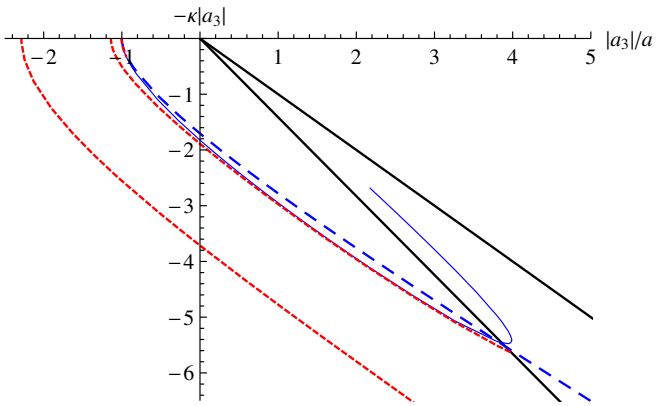

FIG. 2. Expanded view of the three- and four-body bound states with smaller energies. The notation is as in Fig. 1 except for the thin blue line which here shows the effective AT threshold.

The three-body state is bound for atom-atom scattering lengths $|a_3|/a > -1$. Its binding momentum in the unitary limit is $\kappa_3|a_3| = 1.71$, which should be compared with the exact result $\kappa_3|a_3| = 1.5077$ from Deltuva's solutions of the Faddeev equation [24]. Eventually the trimer becomes unbound against decay into the AD channel. It is difficult to determine precisely the point at which this happens in our approach because, as discussed above, the threshold is slightly displaced in our truncation. Since the curves for the dimer and trimer binding energies cross at a very shallow angle, the small error in the threshold position has a disproportionately large effect on the point at which the trimer becomes unbound. Its energy passes through the dimer binding energy at $|a_3|/a \simeq +19$, but does not reach the effective threshold until $|a_3|/a = +31.1$. For comparison, Deltuva's exact solutions give $|a_3|/a = +21.30$ [24, 36].

The tetramer states become bound with respect to the four-atom channel at $|a_3|/a = -2.29$, -1.14 and -1.003 [23]. For comparison, the two states seen in exact treatments appear at $|a_3|/a = -2.351$ and -1.096 [24]. Only two of our states persist to the unitary limit, where their binding momenta are $\kappa|a_3| = 3.71$ and 1.89. Relative to the three-body state, these are $\kappa/\kappa_3 = 2.18$ and 1.11, and so both are overbound compared to the exact results, 2.147 and 1.0011 [10]. The third state is very weakly bound and so is almost invisible in the plots. It is bound only within a very narrow range of atom-atom scattering lengths, hitting the effective AT threshold (which lies slightly below the actual trimer binding energy) at $|a_3|/a = -0.947$.

The deepest tetramer becomes unbound when it reaches the DD threshold at $|a_3|/a = +11.75$, which should be compared with +9.700 from the numbers in Refs. [24, 36]. The second tetramer state remains bound with respect to the trimer until it hits that threshold at $|a_3|/a = +3.99$, to be compared with +3.140. This behaviour contrasts with what is seen in exact treatments [11, 37], where the state is a virtual one over much of the region between $1/a = 0$ and the point where the trimer crosses the dimer-dimer threshold. However it is perhaps worth pointing out that the state lies very close to the effective AT threshold over much of this range.

The energy spectrum shows no sign of the super-Efimov behaviour – an infinite tower of states in a double exponential pattern [38] – seen during the evolution in the regime where the cutoff scale much larger than $1/a$ and γ. This is consistent with the theorem of Amado and Greenwood, which states that there cannot be an infinite number of tetramers with an accumulation point at zero energy [39]. We also see no sign of the AAD Efimov effect predicted by Braaten and Hammer [40] in region where the trimer crosses the DD threshold. This may be another artefact of our truncation since just beyond this point our effective atom-trimer effective AT threshold starts to show unphysical behaviour, bending back on itself instead of following the trimer binding energy (see Fig. 2).

In this work we have extended our previous FRG approach [23] to study the spectrum of three- and four-body states in systems close to the unitary limit. In order to include energy dependence in the three-body sector we introduce a trimer field. We work with a truncated action that retains all energy-independent contact interactions in the four-body sector. We allow the trimer field to run with our cutoff in order to eliminate couplings that involve the AAD "breakup" channel and hence to generate equations with an analogous structure to the Faddeev-Yakubovsky equations.

Our results show similar qualitative behaviour to those obtained from exact treatments of the few-body systems. In many cases we even get quantitative agreement at the 20% level or better. The main exceptions are the length of the Efimov cycle, which is about 30% too large, and the overbinding of the four-body states which leads to the appearance of a third tetramer, if only in a very narrow window. This reinforces the importance of keeping all four-body contact interactions and of including energy dependence in the two- and three-body subsystems.

Improving on these results will require the use of truncations that include more of this energy dependence, though higher-order terms in the propagators and energy- or momentum-dependent couplings. With such improvements, it will be interesting to explore whether the FRG can also describe systems with more particles and strongly interacting bosonic matter.

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